The BESSEL library V1.2

Steven Ahlig

The BESSEL library provides various special functions:

- Bessel functions of the first and of the second kind $J_\nu(x)$ and $Y_\nu(x)$ of integer order ν and for real arguments x.
- modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ of integer order ν and for real arguments x.
- the incomplete gamma function $P(a, x)$ for real a and x.
- the error function $erf(x)$ for real x.
- the beta function $B(a, b)$ for real a, b
- the incomplete beta function $I_x(a, b)$ for real a, b, x
- associated Legendre polynomials $P_{lm}(x)$ for real x
- spherical harmonics $Y_{lm}(\theta, \phi)$ for real arguments θ, ϕ

1. Bessel Functions

The Bessel differential equation (DEQ) can be written in the form

$$\frac{d^2}{dx^2}u(x) + \frac{1}{x}\frac{d}{dx}u(x) + \left(1 - \frac{\nu^2}{x^2}\right)u(x) = 0 \tag{1}$$

This is a linear DEQ, i.e. given that $u_1(x)$ and $u_2(x)$ are solutions of (1) it follows that also $\alpha u_1(x) + \beta u_2(x)$ with $\alpha, \beta \in R$ is a solution; the solutions of (1) thus form a vector space.

The Bessel functions of the first and of the second kind, $J_\nu(x)$ and $Y_\nu(x)$, are solutions of (1) and are defined as follows

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1+\nu)} \left(\frac{x}{2}\right)^{2k} \quad , \tag{2}$$

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \quad . \tag{3}$$

These solutions are linear independent for noninteger $\nu > 0$ and form a basis of the vector space of solutions of (1). However, $J_\nu(x)$ and $Y_\nu(x)$ are linear dependent for $\nu \in N$. A quite tedious calculation reveals that

$$\bar{Y}_\nu(x) = \frac{2}{\pi}J_\nu(x)\log\left(\frac{\gamma x}{2}\right) - \frac{1}{\pi}\sum_{k=0}^{\nu-1}\frac{(\nu-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-\nu} - \frac{1}{\pi}\left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty}(-1)^k \frac{s_k + s_{\nu+k}}{k!(\nu+k)!}\left(\frac{x}{2}\right)^k \tag{4}$$

$$\text{with} \quad s_\nu = \sum_{k=1}^{\nu}\frac{1}{k} \quad \text{and} \quad \gamma = \exp\big(\lim_{\nu\to\infty}(s_\nu - \log(\nu))\big)$$

is the 2nd solution which is linear independent form $J_\nu(x)$ for $\nu \in N$.

Considering the Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ for purely imaginary arguments x one arrives at the modified Bessel functions

$$I_\nu(x) = (-i)^\nu J_\nu(ix) \tag{5}$$

$$K_\nu(x) = \frac{\pi}{2}i^{\nu+1}\big(J_\nu(ix) + iY_\nu(ix)\big) \tag{6}$$

These functions are solutions of the DEQ

$$\frac{d^2}{dx^2}u(x) + \frac{1}{x}\frac{d}{dx}u(x) - \left(1 + \frac{\nu^2}{x^2}\right)u(x) = 0 \quad . \tag{7}$$

Bessel functions have countless applications in all areas of physics and engineering. See, e.g., [1] for applications to boundary value problems in classical electrodynamics and [2] for a 'scholarly work' on the theory of Bessel functions.

The subroutines BESJ, BESY, BESI and BESK check the number and the type of their arguments. Furthermore they do check whether the order n is a positive integer or zero.

The algorithms are taken from [3].

BESJ returns the value of $J_n(x)$ for $n \in N$ and $x \in R$. The stack diagram is

$$\begin{array}{lcl} 2: n & \xrightarrow[\text{BESJ}]{} & 2: \\ 1: x & & 1: J_n(x) \end{array}$$

BESY returns the value of $Y_n(x)$ for $n \in N$ and $x \in R$, $x \geq 0$. The stack diagram is

$$\begin{array}{lcl} 2: n & \xrightarrow[\text{BESY}]{} & 2: \\ 1: x & & 1: Y_n(x) \end{array}$$

$Y_n(0) = -\infty$ is approximated by $Y_n(0) = -\mathrm{MAXREAL}$.
It is checked whether $x \geq 0$.

BESI returns the value of $I_n(x)$ for $n \in N$ and $x \in R$. The stack diagram is

$$\begin{array}{lcl} 2: n & \xrightarrow[\text{BESI}]{} & 2: \\ 1: x & & 1: I_n(x) \end{array}$$

BESK returns the value of $K_n(x)$ for $n \in N$ and $x \in R$, $x \geq 0$. The stack diagram is

$$\begin{array}{lcl} 2: n & \xrightarrow[\text{BESK}]{} & 2: \\ 1: x & & 1: K_n(x) \end{array}$$

$K_n(0) = \infty$ is approximated by $K_n(0) = \mathrm{MAXREAL}$.
It is checked whether $x \geq 0$.

*

The accuracy of the routines may be estimated with the help of the tables 1, 2 and 3 which compare the results of BESJ to corresponding values of $J_n(x)$. BESY, BESI and BESK give results of similar accuracy.

2. Incomplete Gamma Function

The incomplete gamma function P is defined by

$$P(a, x) = \frac{1}{\Gamma(a)} \int_0^x dt \exp(-t) t^{a-1} \quad (a > 0) \tag{8}$$

where Γ is the 'ordinary' gamma function

$$\Gamma(a) = \int_0^\infty dt \exp(-t) t^{a-1} \quad \Rightarrow \quad \Gamma(a+1) = a \cdot \Gamma(a) \quad . \tag{9}$$

Note that Γ is a built-in function of the HP49, it is merged with the factorial command: $\Gamma(x) = (x-1)!$.

The incomplete gamma function (8) encompasses a number of other special functions. A quite probably incomplete list is:

- The error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt \exp(-t^2) \tag{10}$$

is obtained from the incomplete gamma function $P(a,x)$ by the parameter choice $a = 1/2$ and the replacement $x \to x^2$, i.e.

$$\mathrm{erf}(x) = \mathrm{sign}(x) \cdot P\left(\frac{1}{2}, x^2\right) \quad . \tag{11}$$

- The χ^2-probability function is used to quantify the goodness of fit and it is found equal to

$$P\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) \tag{12}$$

where ν is an integer, which stands for the number of degrees of freedom and χ^2 is a measure for the difference of two models or datasets. The quantity $1 - P(\nu/2, \chi^2/2)$ gives the probability that the observed value of chi-square will exceed the value χ^2 by chance even for a correct model. Note that $1 - P(\nu/2, \chi^2/2)$ for integer ν is a built-in function of the HP49, it is called UTPC, more explicitly: $\mathrm{UTPC}(\nu, \chi^2) = 1 - P(\nu/2, \chi^2/2)$. UTPC does accept noninteger ν as argument, however the results of UTPC are wrong for ν not an integer.
- The cumulative Poisson probability function $P_x(< k)$ depends on two variables, x and k. It gives the probability that the number of Poisson random events occuring will be between 0 and $k-1$ inclusive, if the expected mean number is x. One finds $P_x(< k) = 1 - P(k, x)$.

*

The subroutine GAMP checks the number and the type of its arguments as well as the condition $a > 0$. GAMP is based on an algorithm which is taken from [3] and it is used according to the stack diagram

$$\begin{array}{lcl} 2:\ a & \xrightarrow[\text{GAMP}]{} & 2: \\ 1:\ x & & 1:\ P(a,x) \end{array}$$

*

The accuracy of GAMP is illustrated by the tables 4, 5 and 6 where the results of $1-\mathrm{GAMP}(\nu/2, \chi^2/2)$ are compared to corresponding values of the χ^2-probability function, which are taken from [4]. See also table 7 which compares $\mathrm{GAMP}(1/2, x^2)$ to accurate values of the error function.

3. The Error Function

The connection between the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt \exp(-t^2) \tag{13}$$

and the incomplete gamma function has been emphasized in the last section. And with $\Gamma(1/2) = \sqrt{\pi}$ it is easy to convince oneself that indeed $\mathrm{erf}(x) = P(1/2, x^2)$ for $x \geq 0$.

There is also a close relation between $\mathrm{erf}(x)$ and the normal distribution probability function

$$\mathrm{UTPN}(m, \nu, x) = \frac{1}{\sqrt{2\pi\nu}} \int_x^\infty dt \exp\left(-\frac{(t-m)^2}{2\nu}\right) \tag{14}$$

which is a built-in function of the HP49. With

$$\frac{2}{\sqrt{\pi}} \int_0^\infty dt \exp(-t^2) = 1 \tag{15}$$

one finds immediately

$$\mathrm{erf}(x) = 1 - 2 \cdot \mathrm{UTPN}(0, \frac{1}{2}, x) \quad . \tag{16}$$

*

ERF computes the error function $\mathrm{erf}(x)$ for real arguments x. The algorithm implements (16) which gives very accurate results [1]. ERF does check the number and the type of its argument and it is called according to

$$\begin{array}{lcl} 2: & & 2: \\ 1: x & \xrightarrow[\mathrm{ERF}]{} & 1: \mathrm{erf}(x) \end{array}$$

*

Table 7 compares the results of $\mathrm{ERF}(x)$ and $\mathrm{GAMP}(1/2, x^2)$ to corresponding values that are taken from [4].

4. The (incomplete) Beta Function

The Beta function $B(z, w)$ is defined as

$$B(a, b) = \int_0^1 dt\, t^{a-1}(1-t)^{b-1} \quad \text{for } a, b > 0 \tag{17}$$

[1] Thanks to Sascha Haffner for pointing this out to me.

which implies the symmetry $B(a,b) = B(b,a)$. Computationally useful is the relation

$$B(a,b) = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)} \tag{18}$$

which allows efficient calculation of $B(a,b)$ via the Gamma function Γ.

Closely related to $B(a,b)$ is the incomplete Beta function

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x dt\, t^{a-1}(1-t)^{b-1} \quad \text{for } a,b > 0, 0 \leq x \leq 1 \tag{19}$$

This is a function of three arguments, x is usually written as index. Like the incomplete Gamma function, the incomplete Beta function can be related to a number of other special functions, including the hypergeometric function, the F-distribution and the Student t-distribution.

- The Student t-distribution probability function $A(t,\nu)$ is defined as follows:

$$A(t,\nu) = \frac{1}{\sqrt{\nu} B(\frac{1}{2}, \frac{\nu}{2})} \int_{-t}^{t} dy \left(1 + \frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}} \quad 0 \leq t \leq \infty \tag{20}$$

  Assume that X is a normal distributed random variable with mean zero and variance unity, and that χ^2 follows an independent chi-square distribution with ν degrees of freedom. The distribution of the quantity $X/\sqrt{\chi^2\nu}$ is called Student's t-distribution with ν degrees of freedom. $A(t,\nu)$ as defined in (20) gives the probability that $|X/\sqrt{\chi^2\nu}|$ will by chance be less than t.

  This probability function $A(t,\nu)$ has a simple relation to the incomplete Beta function:

$$A(t,\nu) = 1 - I_x(\frac{\nu}{2}, \frac{1}{2}) \quad \text{where } x = \frac{\nu}{\nu + t^2} \tag{21}$$

*

The routines BETA and IBETA calculate $B(a,b)$ and $I_x(a,b)$ respectively. They do check the number and the type of their arguments. It is tested whether $a, b > 0$ and IBETA checks in addition whether $0 \leq x \leq 1$. The algorithm of BETA rests on (18) and the algorithm of IBETA is taken from [3].

$$\begin{array}{ll} 2: & a \\ 1: & b \end{array} \xrightarrow[\text{BETA}]{} \begin{array}{ll} 2: & \\ 1: & B(a,b) \end{array}$$

$$\begin{array}{ll} 3: & a \\ 2: & b \\ 1: & x \end{array} \xrightarrow[\text{IBETA}]{} \begin{array}{ll} 3: & \\ 2: & \\ 1: & I_x(a,b) \end{array}$$

*

The accuracy of BETA is essentially determined by the accuracy of the built-in Gamma function. The tables 8, 9, 10 and 11 are meant to give an idea on the accuracy of IBETA.

5. Associated Legendre Polynomials

The Legendre polynomials $P_n(x), n \in N$ can be defined as

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n \tag{22}$$

which gives

$$P_0(x) = 1 \quad , \quad P_1(x) = x \quad \text{and} \quad P_2(x) = \frac{1}{2}(3x^2-1)\,. \tag{23}$$

These 'ordinary' Legendre polynomials may be used to define the associated Legendre polynomials $P_{lm}(x)$ for positive $m \in N$

$$P_{lm}(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x) \tag{24}$$

and it can be shown that for $m \in Z, m < 0$

$$P_{lm}(x) = (-1)^{|m|} \frac{(l-|m|)!}{(l+|m|)!} P_{l|m|}(x). \tag{25}$$

For every fixed m these associated Legendre polynomials $P_{lm}(x)$ form an orthogonal basis for functions on the interval $]-1,1[$ [2]. That is, a function on $[-1,1]$ can be decomposed as follows

$$g(x) = \sum_{l=0}^{\infty} g^{lm} P_{lm}(x) \quad 0 \le x \le 1 \tag{26}$$

for every fixed $m \in Z$ satisfying $|m| < l$. The coefficients are just numbers and can be extracted by using the orthogonality relation

$$\int_{-1}^{1} dx\, P_{l'm}(x) P_{lm}(x) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{l'l} \tag{27}$$

*

The routine PLM computes the associated Legendre polynomials $P_{lm}(x)$ for real x. Number and type of the arguments are checked and it is tested whether $|m| \le l$. PLM is called according to the stack diagram

$$\begin{array}{lcl} 3:\ l & & 3: \\ 2:\ m & \xrightarrow[\text{PLM}]{} & 2: \\ 1:\ x & & 1:\ P_{lm}(x) \end{array}$$

*

The accuracy of PLM can be read off from the tables 12 and 13.

[2] To be more precise: the $P_{lm}(x)$ form an orthogonal basis for the space $L_2(]-1,1[)$, i.e. for the space of functions $]-1,1[\to R$ that are square integrable.

6. Spherical Harmonics

The spherical harmonics $Y_{lm}(\theta, \phi)$ are defined in terms of the associated Legendre polynomials $P_{lm}(x)$

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_{lm}(\cos(\theta)) \exp(im\phi) \tag{28}$$

where $l \in N$ and $m \in Z, |m| \leq l$. These functions do arise in solving the Laplace equation in spherical coordinates, see e.g. [1].

The set of functions $\{Y_{lm} \,|\, l \in N, m \in Z, |m| \leq l\}$ forms an orthogonal basis for functions on the sphere. That is, a function on the sphere can be decomposed as

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f^{lm} Y_{lm}(\theta, \phi) \tag{29}$$

where the coefficients f^{lm} are numbers, which may be called the components of the function f in the basis $\{Y_{lm}\}$. These coefficients/components f^{lm} can be extracted by using the orthogonality relation

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin(\theta) \bar{Y}_{l'm'}(\theta, \phi) Y_{lm}(\theta, \phi) = \delta_{ll'} \delta_{mm'} \tag{30}$$

where $\bar{Y}_{l'm'}(\theta, \phi)$ denotes the complex conjugate of $Y_{l'm'}(\theta, \phi)$.

The routine YLM computes $Y_{lm}(\theta, \phi)$ for real θ, ϕ [3]. The number and the type of the arguments are checked and it is tested whether $|m| \leq l$. YLM is called according to the stack diagram

$$\begin{array}{lcl} 4: \; l & & 4: \\ 3: \; m & \xrightarrow[\text{YLM}]{} & 3: \\ 2: \; \theta & & 2: \\ 1: \; \phi & & 1: \; Y_{lm}(\theta, \phi) \end{array}$$

*

The algorithm of YLM rests on the corresponding algorithm for the computation of P_{lm} and it is therefore essentially as accurate as PLM.

[3]Note that the result of YLM depends on the angle measure RAD/DEG/GRAD

References

[1] J.D. Jackson, 'Classical Electrodynamics', 2nd edition, John Wiley, 1975

[2] G.N. Watson, 'A Treatise on the Theory of Bessel Functions', Cambridge Univ. Press, 1944

[3] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, 'Numerical Recipes in C', 2nd edition, Cambridge Univ. Press, 1992

[4] M. Abramowitz, I. Stegun, 'Handbook of Mathematical Functions', Dover, 1972

[5] Mathematica 4,V4.0.1.0

Appendix A. Tables

x	$J_0(x)$, [4]	$BESJ(0,x)$
0.1	0.997501562066	0.9975015647
1.5	0.511827671736	0.5118276712
5.0	-0.1775967713	-0.177596774
10.0	-0.2459357644	-0.24593576438

Table 1. Results of $BESJ(0,x)$ compared to values of $J_0(x)$ which are taken from [4]

x	$J_1(x)$, [4]	$BESJ(1,x)$
0.1	0.0499375260	0.04993752604
1.5	0.5579365079	0.55793650789
5.0	-0.3275791376	-0.3275791386
10.0	0.0434727462	0.04347274634

Table 2. Results of $BESJ(1,x)$ compared to values of $J_1(x)$ which are taken from [4]

x	$J_2(x)$, [4]	$BESJ(2,x)$
0.1	0.0012489587	0.00124895866
1.5	0.2320876721	0.23208767218
5.0	0.0465651163	0.0465651187
10.0	0.2546303137	0.25463031365

Table 3. Results of $BESJ(2,x)$ compared to values of $J_2(x)$ which are taken from [4]

$\chi^2 = 0.01$ ν	$1 - P(\nu/2, \chi^2/2)$, [4]	$1 - GAMP(\nu/2, \chi^2/2)$
1	0.92034	0.920344325448
2	0.99501	0.995012479193
3	0.99973	0.999734834941
4	0.99999	0.999987541589

Table 4. Results of $1 - GAMP(\nu/2, \chi^2/2)$ for $\chi^2 = 0.01$ compared to values of $1 - P(\nu/2, \chi^2/2)$ which are taken from [4]. It would be nice, if somebody could give me a hint as to where one can find more accurate values of $P(a,x)$ than those in the 2nd column.

$\chi^2 = 2.2$ ν	$1 - P(\nu/2, \chi^2/2)$, [4]	$1 - GAMP(\nu/2, \chi^2/2)$
1	0.13801	0.138010745551
2	0.33287	0.332871086081
5	0.82084	0.820835969911
10	0.99457	0.994564706554
15	0.99994	0.999944440127

TABLE 5. Results of $1 - GAMP(\nu/2, \chi^2/2)$ for $\chi^2 = 2.2$ compared to values of $1 - P(\nu/2, \chi^2/2)$ which are taken from [4]

$\chi^2 = 5.0$ ν	$1 - P(\nu/2, \chi^2/2)$, [4]	$1 - GAMP(\nu/2, \chi^2/2)$
1	0.02535	0.025347316063
2	0.08209	0.082084998624
5	0.41588	0.415880214401
10	0.89118	0.891178020473
15	0.99213	0.992126411423
20	0.99972	0.999722647909

TABLE 6. Results of $1 - GAMP(\nu/2, \chi^2/2)$ for $\chi^2 = 5.0$ compared to values of $1 - P(\nu/2, \chi^2/2)$ which are taken from [4]

x	$\mathrm{erf}(x)$, [4]	$ERF(x)$	$GAMP(1/2, x^2)$
0.01	0.0112834156	0.011283415556	0.0112834155557
0.02	0.0225645747	0.022564574692	0.0225645746915
0.1	0.1124629160	0.112462916018	0.112462915998
0.5	0.5204998778	0.520499877814	0.520499876065
1.0	0.8427007929	0.84270079295	0.842700790023
1.5	0.9661051465	0.966105146475	0.966105149549

TABLE 7. Results of $ERF(x)$ and $GAMP(1/2, x^2)$ compared to values of $\mathrm{erf}(x)$ which are taken from [4]

a	$\frac{\mathrm{Beta}[a,0.1,0.1]}{\mathrm{Beta}[0.1,0.1]}$, [5]	$\mathrm{IBETA}(a, 0.1, 0.1)$
0.1	0.406385	0.406385093797
0.3	0.133066	0.133066319903
1.0	0.0104807	0.0104807417889
2.0	0.000585549	0.000585549211572

TABLE 8. Results of $\mathrm{IBETA}(x, 0.1, 0.1)$ compared to corresponding results, calculated with [5]. These tables for the incomplete Beta function are not really meant to measure the accuracy of IBETA, since they compare quite probably with a somewhat different implementation of the same algorithm. - I'm still looking for accurate values to compare with.

a	$\frac{\mathrm{Beta}[a,0.1,0.5]}{\mathrm{Beta}(0.1,0.5)}$, [5]	$\mathrm{IBETA}(a, 0.1, 0.5)$
0.1	0.5	0.50000001138
0.3	0.242049	0.24204906039
1.0	0.066967	0.0669670062254
2.0	0.0203154	0.0203153584729

TABLE 9. Results of $\mathrm{IBETA}(x, 0.1, 0.5)$ compared to corresponding results, calculated with [5].

a	$\frac{\mathrm{Beta}[a,1,0.1]}{\mathrm{Beta}(1,0.1)}$, [5]	$\mathrm{IBETA}(a, 1, 0.1)$
0.1	0.794328	0.794328234725
0.3	0.501187	0.501187233629
1.0	0.1	.1
2.0	0.01	0.01

TABLE 10. Results of $\mathrm{IBETA}(x, 1, 0.1)$ compared to corresponding results, calculated with [5].

a	$\frac{\mathrm{Beta}[a,1,0.5]}{\mathrm{Beta}(1,0.5)}$, [5]	$\mathrm{IBETA}(a, 1, 0.5)$
0.1	0.933033	0.933032993775
0.3	0.812252	0.81225240231
1.0	0.5	0.5
2.0	0.25	0.25

TABLE 11. Results of $\mathrm{IBETA}(x, 1, 0.5)$ compared to corresponding results, calculated with [5].

x	$P_{31}(x)$	$PLM(3,1,x)$
0.0	1.5	1.5
0.1	1.41785709788	1.41785709788
0.3	0.786999841172	0.78699984117
0.5	-0.32475952642	-0.324759526425
0.7	-1.55326068321	-1.5532606832
0.9	-1.99419626667	-1.99419626668
1.0	0.0	0.0

TABLE 12. Results of $PLM(3,1,x)$ compared to the direct evaluation of $P_{31}(x) = -\frac{3}{2}\sqrt{1-x^2}(-1+5x^2)$.

x	$P_{74}(x)$	$PLM(7,4,x)$
0.0	0.0	0.0
0.1	-487.33267275	-487.332672753
0.3	-787.64110425	-787.641104247
0.5	121.81640625	121.81640625
0.7	1063.02024675	1063.02024675
0.9	423.85560525	423.85560525
1.0	0.0	0.0

TABLE 13. Results of $PLM(7,4,x)$ compared to the direct evaluation of $P_{74}(x) = \frac{3465}{2}(-3x+19x^3-29x^5+13x^7)$.